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FUNDAMENTAL SOLUTIONS IN PROBLEMS OF BENDING OF ANISOTROPIC PLATES

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Fundamental solutions based on the engineering theory of bending of thin anisotropic plates (Kirchhoff–Love hypotheses) are constructed for anisotropic, in particular, orthotropic plates of canonical plan form (half-plane, quadrant, band, half-band, rectangle, and unbounded plate with an elliptic hole).

Key words: *anisotropic plate, bending, point load, complex potentials, fundamental solution.*

Fundamental solutions (solutions for point loads and dislocations) play an important role in solving problems of elastic extension and bending of anisotropic plates. The fundamental solutions given, solutions of the problem of plates subjected to loads distributed along lines or sites (regions) can be constructed by integration. In the case of bending of anisotropic plates with stress concentrators such as cuts, holes, and cracks, the knowledge of the fundamental solutions allows one to write potential representations at the stress-concentrator contour in the form of the Cauchy-type integrals and solve the boundary-value problem by a numerical method. Moreover, some boundary conditions at the plate contour can be satisfied automatically, which facilitates the numerical implementation. The solutions given below are compared with the known solutions for isotropic plates [1].

1. Let a homogeneous plate made of a material with rectilinear anisotropy (not necessarily orthotropy) occupy the domain $D = \{|x| < \infty,\ |y| < \infty\}$. At the point τ with the coordinates $x = \xi$ and $y = \eta$, the following point loads are applied: the point transverse force P_z and the point bending moments M_x and M_y. According to the classical theory of bending of thin anisotropic plates [2], the Lekhnitskii elastic complex potentials can be written as

$$\Phi_\nu(z_\nu) = E^1_\nu(z_\nu, \tau) + M^1_\nu(z_\nu, \tau), \quad E^1_\nu(z_\nu, \tau) = A_\nu \ln(z_\nu - \tau_\nu), \quad M^1_\nu(z_\nu, \tau) = B_\nu/(z_\nu - \tau_\nu),$$

$$z_\nu = \operatorname{Re} z + \mu_\nu \operatorname{Im} z, \qquad \tau_\nu = \xi + \mu_\nu \eta \qquad (\nu = 1, 2); \tag{1}$$

$$\varphi_\nu(z_\nu) = A_\nu[\ln(z_\nu - \tau_\nu) - 1](z_\nu - \tau_\nu) + B_\nu \ln(z_\nu - \tau_\nu) + D_\nu,$$

$$F_\nu(z_\nu) = A_\nu(z_\nu - \tau_\nu)^2[\ln(z_\nu - \tau_\nu) - 3/2]/2 + B_\nu(z_\nu - \tau_\nu)[\ln(z_\nu - \tau_\nu) - 1] + D_\nu z_\nu + G_\nu,$$

$$F''_\nu(z_\nu) = \Phi_\nu(z_\nu), \qquad \varphi'_\nu(z_\nu) = \Phi_\nu(z_\nu) \qquad (\nu = 1, 2).$$

Here, the term $E^1_\nu(z_\nu, \tau)$ corresponds to the point force P_z and the term $M^1_\nu(z_\nu, \tau)$ to the point bending moment with the components M_x and M_y. The terms $D_\nu z_\nu + G_\nu$ in the expression for $F_\nu(z_\nu)$ determine the displacement of the plate as a rigid body; A_ν and B_ν are unknown complex constants.

The functions $\varphi_\nu(z_\nu)$ and $F_\nu(z_\nu)$ are multivalued functions. Since these functions are expressed in terms of the multivalued function $\ln(z_\nu - \tau_\nu)$, they acquire the following increments upon circulating along an arbitrary loop L around the point $\tau = \xi + i\eta$:

$$\{\varphi_\nu(z_\nu)\}_L = 2\pi i[A_\nu(z_\nu - \tau_\nu) + B_\nu], \qquad \{\Phi_\nu(z_\nu)\} = 2\pi i A_\nu,$$

$$\{F_\nu(z_\nu)\}_L = 2\pi i[A_\nu(z_\nu - \tau_\nu)^2/2 + B_\nu(z_\nu - \tau_\nu)], \qquad \nu = 1, 2.$$

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The equations for determining A_ν and B_ν follow from the condition of single-valuedness of tangential displacements and deflections of the plate ($u + iv$, w) and from the equation of equilibrium of the part of the plate bounded by the contour L:

$$\sum_{\nu=1}^{2}(\mu_\nu^{k-2}A_\nu - \bar\mu_\nu^{k-2}\bar A_\nu) = f_k \quad (k=\overline{1,4}), \qquad f_1 = \frac{P_z}{2\pi i D_{11}}, \quad f_j = 0 \ (j=\overline{2,4}),$$
$$\sum_{\nu=1}^{2}(\mu_\nu^{k-2}B_\nu - \bar\mu_\nu^{k-2}\bar B_\nu) = d_k \quad (k=\overline{1,4}), \qquad d_1 = \frac{M_y}{2\pi i D_{11}}, \quad d_4 = \frac{M_x}{2\pi i D_{22}}, \quad d_3 = d_2 = 0. \tag{2}$$

Here D_{11} and D_{22} are the flexural rigidities of the plate in the x and y directions, respectively.

We denote the angle between the principal direction of orthotropy E_1 and the x axis by φ. For an orthotropic material, we obtain $\mu_{1,2} = \pm\alpha + i\beta$ for $\varphi = 0$. In this case, we have $\operatorname{Im} A_1 = -\operatorname{Im} A_2$ and A_ν is given by the simple expression

$$A_{1,2} = |\mu_1|^2 P_z(\alpha \mp i\beta)/(16\pi\alpha\beta D_{11}).$$

2. We consider an anisotropic half-plane $D = \{x \geqslant 0, |y| < \infty\}$ clamped along the line $x = 0$. Let a point force P_z be applied to the point with the coordinates $x = \xi$ and $y = \eta$. Using the analogy between the plane problem and the problem of plate bending [3], we write the complex potentials

$$\Phi_\nu(z_\nu) = E_\nu^2(z_\nu,\tau) = A_\nu \ln\frac{z_\nu - \tau_\nu}{\mu_\nu} + \bar A_1 l_\nu s_\nu \ln\frac{s_\nu z_\nu - \bar\tau_1}{\bar\mu_1} + \bar A_2 n_\nu m_\nu \ln\frac{m_\nu z_\nu - \bar\tau_2}{\bar\mu_2},$$
$$l_\nu = \frac{\mu_{3-\nu} - \bar\mu_1}{\mu_\nu - \mu_{3-\nu}}, \qquad s_\nu = \frac{\bar\mu_1}{\mu_\nu}, \qquad n_\nu = \frac{\mu_{3-\nu} - \bar\mu_2}{\mu_\nu - \mu_{3-\nu}}, \qquad m_\nu = \frac{\bar\mu_2}{\mu_\nu}, \qquad \nu = 1,2. \tag{3}$$

With accuracy to a complex constant, the first term in (3) is the solution of the problem of an infinite plate loaded by a point force. The second and third terms are functions regular in the domain considered and satisfy the clamped boundary conditions ($w = w_x = 0$) along the line $x = 0$ if the complex constants l_ν and n_ν are chosen properly (a somewhat different form of the solution for an anisotropic clamped half-plane is given in [4]).

For the free edge of the half-plane $x = 0$, the boundary condition yields a system of equations for the constants l_ν and n_ν

$$l_1p_1s_1 + l_2p_2s_2 = -\bar p_1, \qquad l_1s_1^2q_1\mu_1^2 + l_2s_2^2q_2\mu_2^2 = -\bar q_1^2\bar\mu_1^2,$$
$$n_1p_1m_1 + n_2p_2m_2 = -\bar p_2, \qquad n_1m_1^2q_1\mu_1^2 + n_2m_2^2q_2\mu_2^2 = -\bar q_2^2\bar\mu_2^2,$$

which implies that

$$l_\nu = \frac{q_\lambda\bar p_1/\bar\mu_1 - \bar q_1 p_\lambda/\mu_\lambda}{q_\nu p_\lambda/\mu_\lambda - q_\lambda p_\nu/\mu_\nu}, \qquad n_\nu = \frac{q_\lambda\bar p_2/\bar\mu_2 - \bar q_2 p_\lambda/\mu_\lambda}{q_\nu p_\lambda/\mu_\lambda - q_\lambda p_\nu/\mu_\nu}, \qquad \lambda = 3 - \nu. \tag{4}$$

For the simply supported edge ($x = 0$) of the half-plane, the boundary condition yields

$$l_1p_1s_1 + l_2p_2s_2 = -\bar p_1, \qquad l_1s_1^{-1} + l_2s_2^{-1} = -1,$$
$$n_1p_1m_1 + n_2p_2m_2 = -\bar p_2, \qquad n_1m_1^{-1} + n_2m_2^{-1} = -1,$$

and accordingly,

$$l_\nu = \frac{\bar\mu_1 p_\lambda/\mu_\lambda - \mu_\lambda\bar p_1/\bar\mu_1}{\mu_\lambda p_\nu/\mu_\nu - \mu_\nu p_\lambda/\mu_\lambda}, \qquad n_\nu = \frac{\bar\mu_2 p_\lambda/\mu_\lambda - \mu_\lambda\bar p_2/\bar\mu_2}{\mu_\lambda p_\nu/\mu_\nu - \mu_\nu p_\lambda/\mu_\lambda}.$$

We consider an orthotropic ($\varphi = 0$) half-plane $D = \{x \geqslant 0, |y| < \infty\}$ with the simply supported edge $x = 0$. Let two equal but opposite point forces be applied to the infinite plate symmetrically about the y axis. Using the superposition principle, we obtain the complex potentials for a simply supported half-plane bent by a point force

$$\Phi_\nu(z_\nu) = E_\nu^3(z_\nu,\tau) = E_\nu^1(z_\nu,\tau) - E_\nu^1(z_\nu,-\bar\tau) = A_\nu \ln((z_\nu - \tau_\nu)/(z_\nu + \tau_\nu^*)),$$
$$\tau_\nu^* = \xi - \mu_\nu\eta, \quad \nu = 1,2.$$

This formula can be derived from (3) by changing the elastic parameters with allowance for material orthotropy.

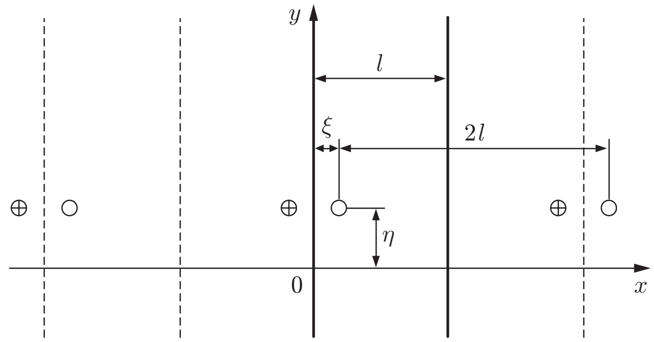

Fig. 1

3. We consider a quadrant (the first quarter of the complex plane) and set $\varphi = 0$. Let the semi-infinite orthotropic plate with a simply supported edge be loaded by oppositely directed point forces symmetrically about the x axis. In this case, we obtain the solution for a point force applied to an internal point of the quadrant simply supported along the sides $x = 0$ and $y = 0$:

$$\Phi_\nu(z_\nu) = E^4_\nu(z_\nu, \tau) = E^1_\nu(z_\nu, \tau) + E^1_\nu(z_\nu, -\tau) - E^1_\nu(z_\nu, \bar{\tau}) - E^1_\nu(z_\nu, -\bar{\tau}),$$

$$\Phi_\nu(z_\nu) = A_\nu \ln \frac{(z_\nu - \tau_\nu)(z_\nu + \tau_\nu)}{(z_\nu + \tau^*_\nu)(z_\nu - \tau^*_\nu)}, \qquad \tau^*_\nu = \xi - \mu_\nu \eta, \qquad \nu = 1, 2.$$

We consider an orthotropic half-plane $D = \{x \geqslant 0,\ |y| < \infty\}$, $\varphi = 0$ clamped along the line $x = 0$. Let two equal but opposite point forces be applied to the orthotropic half-plane symmetrically about the x axis. In this case, we obtain the solution of the problem of bending of an orthotropic quadrant ($x \geqslant 0$, $y \geqslant 0$) clamped along the side $x = 0$ and simply supported along the side $y = 0$. The complex potentials have the form

$$\Phi_\nu(z_\nu) = E^5_\nu(z_\nu, \tau) = E^2_\nu(z_\nu, \tau) - E^2_\nu(z_\nu, \bar{\tau}),$$

$$\Phi_\nu(z_\nu) = A_\nu \ln \frac{z_\nu - \tau_\nu}{z_\nu - \tau^*_\nu} + \bar{A}_1 l_\nu s_\nu \ln \frac{s_\nu z_\nu - \bar{\tau}_1}{s_\nu z_\nu - \bar{\tau}^*_1} + \bar{A}_2 n_\nu m_\nu \ln \frac{m_\nu z_\nu - \bar{\tau}_2}{m_\nu z_\nu - \bar{\tau}^*_2},$$

$$\tau^*_\nu = \xi - \mu_\nu \eta, \qquad \nu = 1, 2.$$

4. Let an anisotropic plane be loaded by a periodic system of point forces applied to a line parallel to the x axis. In this case, one should replace $\ln(z_\nu - \tau_\nu)$ in (1) by $\ln\{\sin[\omega(z_\nu - \tau_{\nu k})]\}$. As a result, the complex potentials become

$$\Phi_\nu(z_\nu) = E^6_\nu(z_\nu, \tau) = A_\nu \ln\{\sin[\omega(z_\nu - \tau_{\nu k})]\},$$
$$\omega = \pi/l, \quad \tau_{\nu k} = \tau_\nu + kl \quad (k = 0, \pm 1, \pm 2, \ldots, \pm\infty). \tag{5}$$

Here l is the period of the applied load.

We consider the orthotropic plane ($\varphi = 0$) loaded by two periodic systems of point forces (Fig. 1). Using the superposition principle and bearing in mind Eq. (5), we obtain the following complex potentials in the problem of bending of an infinite band (of width l) loaded by a point force at an arbitrary point:

$$\Phi_\nu(z_\nu) = E^7_\nu(z_\nu, \tau) = A_\nu \ln\{\sin[\omega(z_\nu - \tau_{\nu k})]/\sin[\omega(z_\nu - t_{\nu k})]\},$$
$$t_{\nu k} = -\xi + \mu_\nu \eta + 2l(k+1), \qquad \tau_{\nu k} = \xi + \mu_\nu \eta + 2kl, \tag{6}$$
$$\omega = \pi/(2l), \quad t_{\nu k} - \tau_{\nu k} = 2(l - \xi) \quad (k = 0, \pm 1, \pm 2, \ldots, \pm\infty).$$

It should be noted that the antiderivative of the function $\Phi_\nu(z_\nu)$ as well as the functions $\varphi_\nu(z_\nu) = F'_\nu(z_\nu)$ in formulas (5) and (6) are expressed in the form of infinite series [1, 5].

5. Let an orthotropic half-band $D = \{0 \leqslant y < \infty,\ 0 \leqslant x \leqslant l\}$, $\varphi = 0$ with simply supported edges be loaded by a point force at the point $\tau = \xi + i\eta$. The corresponding solution can be obtained by summation of the solutions for the system shown in Fig. 2.

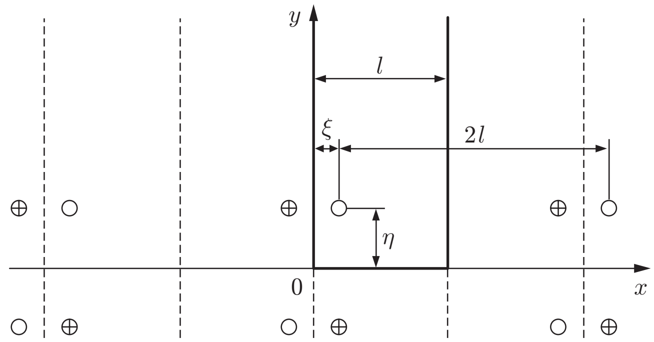

Fig. 2

The complex potentials become

$$\Phi_\nu(z_\nu) = E_\nu^8(z_\nu, \tau) = A_\nu \ln\Big(\frac{\sin[\omega(z_\nu - \tau_{\nu k})]}{\sin[\omega(z_\nu - t_{\nu k})]}\,\frac{\sin[\omega(z_\nu - t^*_{\nu k})]}{\sin[\omega(z_\nu - \tau^*_{\nu k})]}\Big),$$

$$t_{\nu k} = -\xi + \mu_\nu\eta + 2l(k+1), \quad \tau_{\nu k} = \xi + \mu_\nu\eta + 2kl, \quad t^*_{\nu k} = -\xi - \mu_\nu\eta + 2l(k+1), \quad \tau^*_{\nu k} = \xi - \mu_\nu\eta + 2kl,$$

$$\omega = \pi/(2l), \qquad t_{\nu k} - \tau_{\nu k} = t^*_{\nu k} - \tau^*_{\nu k} = 2(l-\xi) \qquad (k = 0, \pm1, \pm2, \ldots, \pm\infty).$$

Given the solution for a simply supported half-band, one can obtain the solution for an orthotropic ($\varphi = 0$) rectangular plate with simply supported edges. To this end, it is necessary to apply a load to the band (Fig. 3). In this case, the complex potentials are written as an infinite series

$$\Phi_\nu(z_\nu) = E_\nu^9(z_\nu, \tau) = \sum_{n=0}^{\infty} A_\nu \ln\Big(\frac{\sin[\omega(z_\nu - \tau_{\nu kn})]}{\sin[\omega(z_\nu - t_{\nu kn})]}\,\frac{\sin[\omega(z_\nu - t^*_{\nu kn})]}{\sin[\omega(z_\nu - \tau^*_{\nu kn})]}\Big),$$

$$t_{\nu kn} = -\xi + \mu_\nu(\eta + n2l_1) + 2l(k+1), \qquad \tau_{\nu kn} = \xi + \mu_\nu(\eta + n2l_1) + 2kl,$$

$$t^*_{\nu kn} = -\xi - \mu_\nu(\eta + n2l_1) + 2l(k+1), \qquad \tau^*_{\nu kn} = \xi - \mu_\nu(\eta + n2l_1) + 2kl,$$

$$\omega = \pi/(2l), \qquad t_{\nu kn} - \tau_{\nu kn} = t^*_{\nu kn} - \tau^*_{\nu kn} = 2(l-\xi)$$

$$(k = 0, \pm1, \pm2, \ldots, \pm\infty, \qquad n = 0, 1, 2, 3, \ldots, \infty).$$

This series converges rapidly; hence, three or four first terms suffice to determine the static or kinematic quantities.

6. Let us consider an orthotropic half-band $D = \{0 \leqslant y < \infty,\ 0 \leqslant x \leqslant l\}$, $\varphi = 0$ whose semi-infinite edges $x = 0, l$ are simply supported and the side $y = 0$ of length l is clamped. The solution of the problem of this plate loaded by a point force can be obtained by summation of the solution of the system shown in Fig. 4. For the upper half-plane $y \geqslant 0$, the complex potentials (3) become

$$\Phi_\nu(z_\nu) = A_\nu \ln(z_\nu - \tau_\nu) + \bar{A}_1 l_\nu \ln(z_\nu - \bar{\tau}_1) + \bar{A}_2 n_\nu \ln(z_\nu - \bar{\tau}_2).$$

The solution for the upper half-plane loaded by a periodic system of point forces can be written as

$$\Phi_\nu(z_\nu) = E_\nu^{10}(z_\nu, \tau) = A_\nu \ln\{\sin[\omega(z_\nu - \tau_{\nu k})]\} + \bar{A}_1 l_\nu \ln\{\sin[\omega(z_\nu - \bar{\tau}_{1k})]\} + \bar{A}_2 n_\nu \ln\{\sin[\omega(z_\nu - \bar{\tau}_{2k})]\},$$

$$\omega = \pi/(2l), \qquad \tau_{\nu k} = \xi + \mu_\nu\eta + 2lk.$$

For the half-band (Fig. 4), the complex potentials become

$$\Phi_\nu(z_\nu) = E_\nu^{11}(z_\nu, \tau) = E_\nu^{10}(z_\nu, \tau) - E_\nu^{10}(z_\nu, t)$$

$$= A_\nu \ln\frac{\sin[\omega(z_\nu - \tau_{\nu k})]}{\sin[\omega(z_\nu - t_{\nu k})]} + \bar{A}_1 l_\nu \ln\frac{\sin[\omega(z_\nu - \bar{\tau}_{1k})]}{\sin[\omega(z_\nu - \bar{t}_{1k})]} + \bar{A}_2 n_\nu \ln\frac{\sin[\omega(z_\nu - \bar{\tau}_{2k})]}{\sin[\omega(z_\nu - \bar{t}_{2k})]}, \tag{7}$$

$$\omega = \pi/(2l), \qquad \tau_{\nu k} = \xi + \mu_\nu\eta + 2lk, \qquad t_{\nu k} = -\xi + \mu_\nu\eta + 2l(k+1) \qquad (k = 0, \pm1, \pm2, \pm3, \ldots, \pm\infty).$$

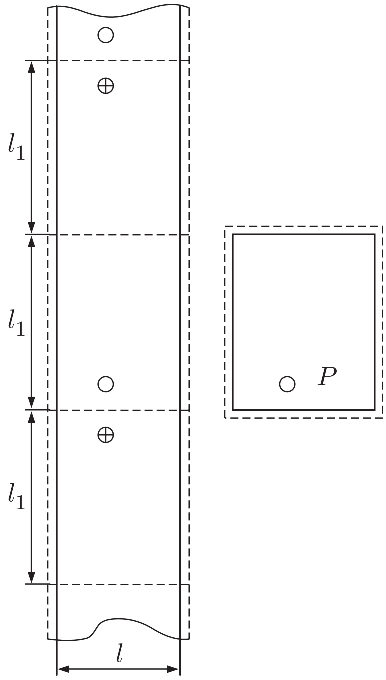

Fig. 3

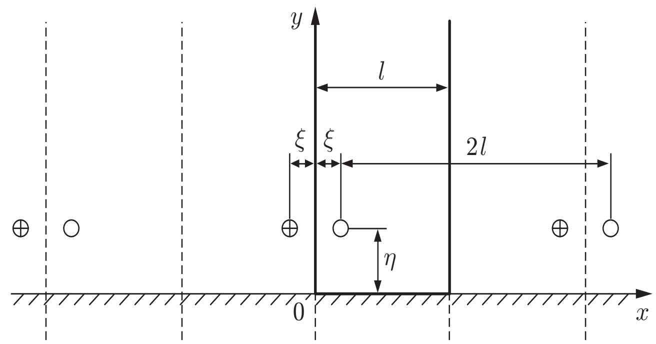

Fig. 4

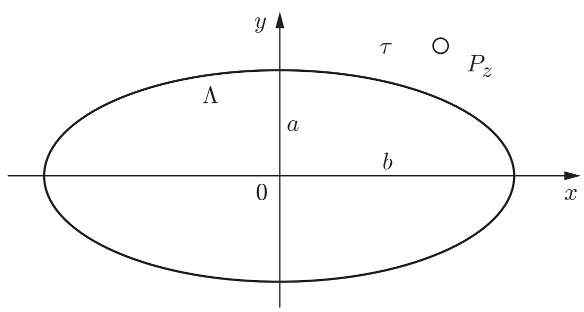

Fig. 5

Using the complex potentials obtained above, by analogy with the clamped half-plane, one can obtain the fundamental solutions for an orthotropic quadrant with free and simply supported edges and the solution for a half-band with a free finite edge and simply supported semi-infinite edges. The relations for a simply supported orthotropic half-band can be obtained directly from (7) by replacing the elastic parameters with allowance for orthotropy (for the corresponding values of l_ν and n_ν).

7. The fundamental solutions obtained above can be used to construct complex potentials (for the domains mentioned above) in the case of a pair of point forces with a unit moment $M = \exp(i\psi)$ applied at the point $\tau = \xi + i\eta$:

$$\Phi_\nu(z_\nu) = M_\nu^k(z_\nu, \tau, \psi) = \frac{d}{ds}\,[E_\nu^k(z_\nu, \tau + s\exp(i\psi))]_{s=0}, \qquad \nu = 1, 2.$$

In this case, the complex constants A_ν are replaced by B_ν:

$$B_\nu = A_\nu H(\psi), \qquad H(\psi) = \cos\psi + \mu_\nu \sin\psi.$$

8. We consider an infinite anisotropic plate with an elliptic hole Λ (Fig. 5). The coordinate origin is located at the center of the ellipse, and the x and y axes coincide with the axes of the ellipse (a and b are the semiaxes of the ellipse). Using the procedure proposed by Grilitskii [6] and taking into account the analogy between the plane problem and the problem of plate bending, we obtain the solution of the problem of a plate with an elliptic hole loaded by a point force:

$$\Phi(z_\nu) = E_\nu^{12}(z_\nu, \tau) = \omega_\nu'(\zeta_\nu)^{-1}[A_\nu \Psi_\nu(\zeta_\nu) \ln(\zeta_\nu - \eta_\nu)$$

$$+ l_\nu \bar{A}_1 \Psi_\nu^1(\zeta_\nu) \ln(\zeta_\nu^{-1} - \bar{\eta}_1) + n_\nu \bar{A}_2 \Psi_\nu^2(\zeta_\nu) \ln(\zeta_\nu^{-1} - \bar{\eta}_2) + \Psi_\nu^3(\zeta_\nu, \eta_\nu)]. \quad (8)$$

In deriving this relation, we used the conformal mappings of the exterior of a unit circle $\gamma = |\sigma| = 1$ onto the exterior of the elliptic holes Λ_ν in the planes $z_\nu = x + \mu_\nu y$ and Λ (in the plane $z = x + iy$):

$$z_\nu = \omega_\nu(\zeta_\nu) = \frac{a - i\mu_\nu b}{2} \zeta_\nu + \frac{a + i\mu_\nu b}{2} \frac{1}{\zeta_\nu}, \qquad z = \omega(\zeta) = \frac{a+b}{2} \zeta + \frac{a-b}{2} \frac{1}{\zeta} \quad (9)$$

and the inverse functions

$$\zeta_\nu = \zeta_\nu(z_\nu) = \frac{z_\nu - \sqrt{z_\nu^2 - (a^2 + \mu_\nu^2 b^2)}}{a - i\mu_\nu b}, \quad \zeta = \zeta(z) = \frac{z - \sqrt{z^2 - a^2 + b^2}}{a + b}, \quad \eta_\nu = \zeta_\nu(\tau_\nu). \quad (10)$$

The functions $\Psi_\nu(\zeta_\nu)$, $\Psi_\nu^i(\zeta_\nu)$ $(i = 1, 2)$, and $\Psi_\nu^3(\zeta_\nu, \eta_\nu)$ are analytical outside the unit circle.

If a point bending moment is applied to the plate, the complex potentials become

$$\Phi_\nu(z_\nu) = M_\nu^{12}(z_\nu, \tau) = \frac{1}{\omega_\nu'(\zeta_\nu)} \Big(\frac{B_\nu}{\zeta_\nu - \eta_\nu} + \frac{l_\nu \bar{B}_1}{\zeta_\nu(\zeta_\nu \bar{\eta}_1 - 1)} + \frac{n_\nu \bar{B}_2}{\zeta_\nu(\zeta_\nu \bar{\eta}_2 - 1)} \Big). \quad (11)$$

The boundary conditions at the contour of the hole Λ are determined by the values of l_ν and n_ν. If the hole contour is clamped or free, the quantities l_ν and n_ν should be found from (3) or (4), respectively. The complex constants B_ν are determined above (see Sec. 7).

It should be noted that the problem cannot be solved if mixed boundary conditions ($w = 0$ and $M_n = 0$) are specified at the contour of the elliptic hole, because the boundary conditions cannot be written in terms of the functions $\varphi_\nu(z_\nu)$. This is possible in the case of kinematic or static boundary conditions [2].

9. The solution of the problem of a plate with a rectilinear cut under point loads can be obtained from (8)–(11) by setting $b = 0$. In particular, for a point moment we obtain

$$\Phi(z_\nu) = M_\nu^{13}(z_\nu, \tau) = \frac{B_\nu}{z_\nu - \tau_\nu} - \frac{B_\nu[I(z_\nu) - I(\tau_\nu)]}{2\sqrt{z_\nu^2 - a^2}\,(z_\nu - \tau_\nu)}$$

$$+ \frac{\bar{B}_1 l_\nu [I(z_\nu) - I(\bar{\tau}_1)]}{2(z_\nu - \bar{\tau}_1)} + \frac{\bar{B}_2 n_\nu [I(z_\nu) - I(\bar{\tau}_2)]}{2(z_\nu - \bar{\tau}_2)},$$

$$I(z) = \sqrt{z^2 - a^2} - z.$$

Changing the variables $z' = z + a$ and $\tau' = \tau + a$, from the last expression we obtain the solution of the problem of a plate with a rectilinear cut along the real axis $0 < x < 2a$ loaded by a point moment. Passing to the limit as $a \to \infty$, we find the solution of the problem of a plate with a semi-infinite cut along the real axis sector $L = \{0 \leqslant x \leqslant \infty, y = 0\}$ subjected to a point moment:

$$\Phi(z_\nu) = M_\nu^{14}(z_\nu, \tau) = \frac{B_\nu}{z_\nu - \tau_\nu} - \frac{B_\nu}{2} \frac{1}{\sqrt{z_\nu}\,(\sqrt{z_\nu} + \sqrt{\tau_\nu}\,)}$$

$$+ \frac{l_\nu}{2} \frac{\bar{B}_1}{\sqrt{z_\nu}\,(\sqrt{z_\nu} + \sqrt{\bar{\tau}_1}\,)} + \frac{n_\nu}{2} \frac{\bar{B}_2}{\sqrt{z_\nu}\,(\sqrt{z_\nu} + \sqrt{\bar{\tau}_2}\,)}.$$

10. The fundamental solutions (Green functions) play an important role in solving problems of bending of plates in the presence of point dislocations. Given these solutions, one can construct potential representations in the form of integrals of displacement discontinuities distributed with unknown density, which ensure specified displacement discontinuities along open or closed curves [7]. To determine the discontinuity density, one should use conditions at the defect which yield one integral equation or a system of integral equations.

Let the plate contain dislocations that can be considered as discontinuities in displacements (deflections and related tangential displacements). If the deflection w has a discontinuity, the stresses in the plate are determined by formulas corresponding to the case of the point force P_z; however, the constants A_ν should be determined from the first system (2) in which the right-side vector-column $\boldsymbol{f}_j$ $(j = \overline{1,4})$ has the components $f_1 = w/(2\pi i)$ and $f_2 = f_3 = f_4 = 0$. If the increments in tangential displacements $w_x + iw_y$ are specified, one should set $d_1 = w_x/(2\pi i)$, $d_2 = w_y/(2\pi i)$, and $d_3 = d_4 = 0$ in the right-side vector-column $\boldsymbol{d}_j$ in (2) to determine B_ν. In this case, the stresses and strains should be determined by formulas that describe the action of a point bending moment.

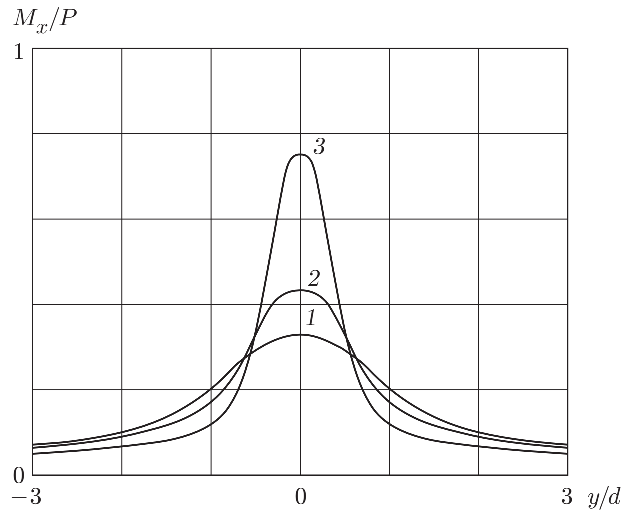

Fig. 6

TABLE 1

Material No.	$E_1 \cdot 10^{-4}$, MPa	$E_2 \cdot 10^{-4}$, MPa	E_1/E_2	$G \cdot 10^{-4}$, MPa	$M_{x\,\max}/P_z$
1	27.610	27.610	1	11.044	0.3183
2	5.384	1.795	3	0.863	0.4398
3	27.610	1.104	25	0.552	0.7669

11. Some capabilities of the potential representations proposed are illustrated in Fig. 6, which shows the distribution of the stresses M_x over the cross section $x = 0$ of a half-plane (along the clamped edge) for various orthotropic materials whose characteristics are listed in Table 1 (the numbers at the curves are the material numbers). For all the materials considered, Poisson's ratio is $\nu_1 = 0.25$. With an increase in the degree of orthotropy E_1/E_2, the maximum stresses $M_{x\,\max}$ at the clamped edge increase (see Table 1). At the same time, the stresses decay more rapidly with distance from the projection of the point where the force P_z is applied onto the line $x = 0$. In the case of an orthotropic half-plane, one can obtain the following expression for the stresses along the clamped boundary:

$$M_x = \frac{|\mu_1|^2 P_z}{2\pi\alpha}\Big(\arctan\frac{(y-\eta)(\alpha^2+\beta^2)-\xi\alpha}{\xi\beta} - \arctan\frac{(y-\eta)(\alpha^2+\beta^2)+\xi\alpha}{\xi\beta}\Big).$$

For an isotropic material ($M_{x\,\max} = P_z/\pi$), this result can be obtained from either solution (3) or the latter formula by using the limiting values of the anisotropy parameters ($\alpha \to 0$ and $\beta \to 1$).

REFERENCES

1. Z. Konchkovskii, *Plates. Static Calculations* [in Russian], Stroizdat, Moscow (1984).
2. S. G. Lekhnitskii, *Anisotropic Plates*, Gordon and Breach, New York (1968)
3. V. N. Maksimenko and E. G. Podruzhin, "Bending of anisotropic plates with arbitrarily shaped cracks," *Uch. Zap. TsAGI*, **20**, No. 3, 81–90 (1989).
4. L. A. Fil'shtinskii and V. A. Lyubchak, "Bending of a semi-infinite anisotropic plate weakened by curvilinear cuts," *Prikl. Mekh.*, **18**, No. 10, 63–67 (1982).
5. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Company, New York (1968).
6. D. V. Grilitskii, "Effect of the point of application of a force or moment on the stress distribution in an infinite anisotropic plate with an elliptic hole," *Prikl. Mekh.*, **2**, No. 2, 159–166 (1956).
7. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Leyden (1977).